Cyclic pointed fusion categories

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1 Preliminaries

The goal of this work is to describe pointed fusion categories with underlying cyclic group.

By a *tensor category* we mean a locally finite rigid $\mathbb{C}$-linear abelian monoidal category such that the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilineal on morphisms and the unit object $\mathbf{1}$ is simple. In particular, in a tensor category $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq \mathbb{C}$. A *fusion category* is a semisimple tensor category with finitely many isomorphism classes of simple objects.

If all simple objects in a fusion category are invertible then it is called *pointed fusion category*. Pointed fusion categories are equivalent to fusion categories Vect_G^{ω} of finite dimensional vector spaces graded by a finite group G, with associativity isomorphism determined by a 3-cocycle ω.

The main result in this work gives a classification for categories of the form $\mathrm{Vect}_{\mathbb{Z}_m}^{\omega}$ up to equivalence, where $\mathbb{Z}_m$ denotes the cyclic group of order $m \in \mathbb{N}$.

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On this section, let $\mathcal{C}$ be a tensor category and X an object in $\mathcal{C}$ such that there exists an isomorphism $\lambda : X^{\otimes n} \xrightarrow{\sim} \mathbf{1}$. Note that there exists a constant $\xi \in \mathbb{K}$ such that the diagram

$$\begin{array}{ccc} X^{\otimes(n+1)} & \xrightarrow{\xi(\mathrm{id}\otimes\lambda)} & X \otimes \mathbf{1} \\ {\scriptstyle \lambda\otimes\mathrm{id}}\downarrow & & \downarrow \\ \mathbf{1} \otimes X & \longrightarrow & X \end{array} \tag{1}$$

commutes. From now on we will call this constant *the constant associated to* X, and we denote it by ξ_X. A priory, said constant depends on the choice of isomorphism $X^{\otimes n} \xrightarrow{\sim} \mathbf{1}$. We have the following lemma.

Lemma 2.1. *The constant associated to X does not depend on the choice of isomorphism $X \xrightarrow{\sim} \mathbf{1}$.*

Proof. Let λ and ρ be isomorphisms $X^{\otimes n} \xrightarrow{\sim} \mathbf{1}$ and $\xi_\lambda, \xi_\rho \in \mathbb{C}$ such that

$$\lambda \otimes \mathrm{id} = \xi_\lambda(\mathrm{id} \otimes \lambda),$$
$$\rho \otimes \mathrm{id} = \xi_\rho(\mathrm{id} \otimes \rho).$$

We show that $\xi_\lambda = \xi_\rho$. In fact, by Schur's Lemma there exists $\alpha \in \mathbb{K}$ such that $\lambda = \alpha\rho$ and thus

$$\rho \otimes \mathrm{id} = \xi_\rho(\mathrm{id} \otimes \rho) = \xi_\rho \alpha^{-1}(\mathrm{id} \otimes \lambda) = \xi_\rho \xi_\lambda^{-1} \alpha^{-1}(\lambda \otimes \mathrm{id}) = \xi_\rho \xi_\lambda^{-1}(\rho \otimes \mathrm{id}).$$

Hence $\xi_\lambda = \xi_\rho$. □

Lemma 2.2. *The constant associated to X is an n^{th} root of unity.*

Proof. Fix $\lambda : X^{\otimes n} \xrightarrow{\sim} \mathbf{1}$. We are going to compute $\lambda \otimes \lambda$ in two different ways. By functoriality of the tensor product, $\lambda \otimes \lambda = \lambda\, (\lambda \otimes \mathrm{id}^{\otimes n})$. We show by induction on k that $\lambda \otimes \lambda = \xi^k \lambda(\mathrm{id}^{\otimes k} \otimes \lambda \otimes \mathrm{id}^{\otimes(n-k)})$ for all $1 \le k \le n$. Since the diagram (1) commutes, we get that

$$\lambda \otimes \lambda = \lambda(\lambda \otimes \mathrm{id}^{\otimes(n)}) = \xi\lambda(\mathrm{id} \otimes \lambda \otimes \mathrm{id}^{\otimes n-1}).$$

and thus the claim is true for $k = 1$. Assume that $\lambda \otimes \lambda = \xi^k \lambda(\mathrm{id}^{\otimes k} \otimes \lambda \otimes \mathrm{id}^{\otimes(n-k)})$ for $1 \le k < n$. Applying diagram (1) to $\mathrm{id}^{\otimes k} \otimes \lambda \otimes \mathrm{id}^{\otimes(n-k)}$ we get that

$$\mathrm{id}^{\otimes k} \otimes \lambda \otimes \mathrm{id}^{\otimes(n-k)} = \xi\, \mathrm{id}^{\otimes k+1} \otimes \lambda \otimes \mathrm{id}^{\otimes(n-k-1)}$$

and thus by this and the inductive hypothesis

$$\lambda \otimes \lambda = \xi^k \lambda(\mathrm{id}^{\otimes k} \otimes \lambda \otimes \mathrm{id}^{\otimes(n-k)}) = \xi^{k+1} \lambda(\mathrm{id}^{\otimes k+1} \otimes \lambda \otimes \mathrm{id}^{\otimes(n-k-1)}),$$

which is what we wanted. In particular, this implies that

$$\lambda \otimes \lambda = \xi^n \lambda(\mathrm{id}^{\otimes n} \otimes \lambda). \tag{2}$$

On the other hand, due to the functoriality of the tensor product we have that

$$\lambda \otimes \lambda = \lambda(\mathrm{id}^{\otimes n} \otimes \lambda). \tag{3}$$

Hence by equations (2) and (3) we conclude $\xi^n = 1$. □

Consider now the object $X^{\otimes j} \in \mathcal{C}$ for some $j \in \mathbb{N}$. Fix an isomorphism $\lambda : X^{\otimes n} \xrightarrow{\sim} \mathbf{1}$. Then we have an isomorphism $\lambda^{\otimes j} : X^{nj} \xrightarrow{\sim} \mathbf{1}$ and it makes sense to compute the constant associated to $X^{\otimes j}$. We arrive to the following result.

Lemma 2.3. *The constant associated to $X^{\otimes j}$ is exactly ξ^{j^2}.*

Proof. Note that by diagram (1)

$$\mathrm{id}_X^{\otimes j} \otimes \lambda^{\otimes j} = \xi^j\ \lambda \otimes \mathrm{id}_X^{\otimes j} \otimes \lambda^{\otimes(j-1)}.$$

Repeating the previous step $j-1$ more times we get

$$\mathrm{id}_X^{\otimes j} \otimes \lambda^{\otimes j} = \xi^{j^2} \lambda^{\otimes j} \otimes \mathrm{id}_X^j\,.$$

Hence $\xi_{X^j} = \xi_X^{j^2}$. □

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Let n be a natural number and let $\mathbb{Z}_n$ be the cyclic group of order n. Let $\zeta \in \mathbb{K}$ be an nth root of 1. Then ζ determines a 3-cocyle ω_ζ on $\mathbb{Z}_n$ in the following way

$$\omega_\zeta(i,j,k) = \zeta^{\frac{i(j+k-(j+k)')}{n}},$$

where for an integer m we denote by m' the remainder of the division of m by n. Moreover, all 3-cocyles in $\mathbb{Z}_n$ modulo coboundaries are of the form ω_ζ for some nth root of unity ζ (see [EGNO, Example 2.6.4]). We denote by $\mathrm{Vect}_{\mathbb{Z}_n}^{\zeta}$ the pointed fusion category corresponding to the 3-cycycle ω_ζ .

For any generator X in the category $\mathrm{Vect}_{\mathbb{Z}_n}^{\zeta}$ there exists an isomorphism $X^{\otimes n} \simeq \mathbf{1}$. Hence it makes sense to wonder if the constant associated to a generator X is an invariant in the category.

Lemma 3.1. *The constant associated to a generator X in $\mathrm{Vect}_{\mathbb{Z}_n}^{\zeta}$ is ζ.*

Proof. Let X be a generator and consider λ to be the canonical isomorphism $X^{\otimes n} \simeq \mathbf{1}$. Note that in this case the constant ξ for which the diagram

$$\begin{array}{ccc} X^{\otimes(n+1)} & \xrightarrow{\xi(\mathrm{id}\otimes\lambda)} & X\otimes\mathbf{1} \\ {\scriptstyle \lambda\otimes\mathrm{id}}\downarrow & & \downarrow \\ \mathbf{1}\otimes X & \longrightarrow & X \end{array}$$

commutes is given by the associativity map from $X^{\otimes n} \otimes X$ to $X \otimes X^{\otimes n}$. That is,

$$\xi = \prod_{k=1}^{n-1} \omega_\zeta(1,k,1) = \prod_{k=1}^{n-1} \zeta^{\frac{(k+1-(k+1)')}{n}} = \zeta.$$

□

Corollary 3.2. *The categories $\mathrm{Vect}_{\mathbb{Z}_m}^{\xi}$ and $\mathrm{Vect}_{\mathbb{Z}_m}^{\zeta}$ are equivalent if and only if there exists $j \in \{1, \cdots, n\}$ such that $\gcd(j,n) = 1$ and $\xi^{j^2} = \zeta$.*

Proof. The previous result implies that the constant associated to generators is in fact an invariant of $\mathrm{Vect}_{\mathbb{Z}_n}^{\zeta}$. Fix a generator X in the category $\mathrm{Vect}_{\mathbb{Z}_n}^{\zeta}$. Note that for every $j \in \{1, \cdots, n\}$ such that $\gcd(j,n) = 1$ we get that $X^{\otimes j}$ is a also a generator in $\mathrm{Vect}_{\mathbb{Z}_n}^{\zeta}$. If ξ denotes the constant associated to X, by Lemma 2.3 the constant associated to $X^{\otimes j}$ is ξ^{j^2}. The result follows. □

Theorem 3.3. *Let $m \in \mathbb{N}$ and let $p_1, \cdots, p_k$ be odd distinct primes such that $m = 2^{n_0} p_1^{n_1} \cdots p_k^{n_k}$ for some $n_0, \cdots, n_k \in \mathbb{N}$. Then there are $a(m)$ categories of* *the form* $\mathrm{Vect}_{\mathbb{Z}_m}^{\xi}$ *up to equivalence, where*

$$a(m) = \begin{cases} \prod_{i=1}^{k}(2n_i+1) & \text{if } n_0 = 0 \\ 2\prod_{i=1}^{k}(2n_i+1) & \text{if } n_0 = 1 \\ 4\prod_{i=1}^{k}(2n_i+1) & \text{if } n_0 = 2 \\ 4(n_0-1)\prod_{i=1}^{k}(2n_i+1) & \text{if } n_0 \geq 3. \end{cases}$$

Proof. First, assume $m = p^k$ for some prime p and $k \in \mathbb{N}$. Consider the action

$$\begin{aligned} (Z_{p^k})^{\times} &\to \mathrm{End}(\mathbb{Z}_{p^k}) \\ l &\mapsto (a \mapsto l^2 a). \end{aligned}$$

By the previous remark the statement reduces to computing the amount of orbits of this action.

Let $a, b \in \mathbb{Z}_{p^k}$. Note that a and b are in the same orbit if and only if there exists $x \in (Z_{p^k})^{\times}$ such that $a \equiv b \bmod p^k$. Hence a and b are in the same orbit there exists $y \in (Z_{p^k})^{\times}$ such that $a \equiv b \bmod p^k$, which is equivalent to $\gcd(a, p^k) = (b, p^k) = p^l$ for some $l < m$.

Define the equivalence classes $H_0, \cdots, H_m$ in $\mathbb{Z}_{p^k}$ where for $x \in \mathbb{Z}_{p^k}$ we have that $x \in H_i$ if and only if x is divisible by p^i but not by p^{i+1}. Then it is enough to look at the orbits inside each class. Fix $i < k$. Note that any element in H_i can be written as yp^i for some $y \in (\mathbb{Z}_{p^k})^{\times}$. Let $y_1p^i, y_2p^i \in H_i$, where $y_1, y_2 \in (\mathbb{Z}_{p^k})^{\times}$. Then y_1p^i, y_2p^i are in the same orbit if and only if there exists $x \in (Z^{p^k})^{\times}$ such that

$$y_1 p^i \equiv x^2 y_2 p^i \bmod p^k. \tag{4}$$

That is

$$y_1 \equiv x^2 y_2 \bmod p^{k-i}$$

which is equivalent to

$$y_1 y_2^{-1} \equiv x^2 \bmod p^{k-i}.$$

Hence if $\mathcal{G}_{k-i}$ is the subgroup of quadratic residues of $\mathbb{Z}_{p^{k-i}}$ this implies that the amount of orbits of the action in H_i is exactly

$$\left|\mathbb{Z}_{p^{k-i}}/\mathcal{G}_{k-i}\right|$$

for all $0 \leq i < k$. If p is odd, $\left|\mathbb{Z}_{p^{k-i}}/\mathcal{G}_{k-i}\right| = 2$ for all $0 \leq i < k$, and this together with the fact that H_k has only one orbit implies that the total amount of orbits of this action is $2k+1$. On the other hand, if $p=2$ then

$$\left|\mathbb{Z}_{p^{k-i}}/\mathcal{G}_{k-i}\right| = \begin{cases} 1 & \text{if } i = k-1 \\ 2 & \text{if } i = k-2 \\ 4 & \text{if } 0 \leq i \leq k-3. \end{cases}$$

Hence if $k=1$ the action has exactly two orbits, if $k=2$ the action has four orbits and if $k \geq 3$ the action has $4(k-1)$ orbits.

Finally, for $m = 2^{n_0} p_1^{n_1} \cdots p_k^{n_k}$ note that the action

$$\begin{aligned} (Z_m)^\times &\to \mathrm{End}(\mathbb{Z}_m) \\ l &\mapsto (a \mapsto l^2 a). \end{aligned}$$

preserves the decomposition

$$\mathbb{Z}_m \simeq \mathbb{Z}_{2^{n_0}} \times \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$$

and thus the amount of orbits of the action on $\mathbb{Z}_m$ is exactly the product of the amount of orbits of the action restricted to each of the terms $\mathbb{Z}_{p_i^{n_i}}$. The result follows. □

Example 3.4. *We compute the number of equivalence classes $a(m)$ of categories of the form $\mathrm{Vect}_{\mathbb{Z}_m}^{\xi}$ for $m \in \{1, \ldots, 10\}$. We have that*

$$\begin{aligned} a(1) &= 1, \\ a(2) &= 2, \\ a(3) &= 3, \\ a(4) &= 4, \\ a(5) &= 3, \\ a(6) &= 6, \\ a(7) &= 3, \\ a(8) &= 8, \\ a(9) &= 5, \\ a(10) &= 6. \end{aligned}$$

References

[EGNO] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*. Math. Surv. Monog. **205**, Amer. Math. Soc.(2015).